Finite-time model matching for linear systems

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Introduction

- Model matching:
  Given a plant $T(s)$ and a matching model $T_m(s)$, we want to find a pre-compensator $C(s)$, so that the transfer function

$$T(s)C(s) = T_m(s)$$

- Applications
  - Tracking
  - Stable inversion
  - Design of desired closed-loop dynamics
  - ...

Plan presentation:

1. Part.1 Algebraic preliminaries
2. Part. 2 Model matching problem
3. Part. 3 Model matching for scalar linear systems
   - Procedure
   - Example for model matching
   - Example for stable inversion (application)
4. Part. 4 Model matching for multivariable linear systems
   - Procedure
   - Example for model matching
   - Example for stable inversion (application)
5. Part.5 Conclusion

Distributed delay and Several rings

Distributed delay

Define $\mathscr{K}(\mathbb{I}_{a,b})$ as the set of real valued functions $g(\cdot)$ of the form

$$g(t) = \begin{cases} g_{\mathbb{I}_{a,b}}(t) & , \quad t \in \mathbb{I}_{a,b} \\ 0 & , \quad \text{elsewhere} \end{cases} \quad g_{\mathbb{I}_{a,b}}(t) = \sum_{i \geq 0} \sum_{j \geq 0} c_{ij} t^j e^{\lambda_i t}$$

Distributed delay: Input-output convolution operator

$$y(t) = (g * u)(t) = \int_{h_1}^{h_2} g_{\mathbb{I}_{h_1,h_2}}(\tau) u(t-\tau) \mathrm{d}\tau$$

where the kernel $g = \sum_{i \geq 0} \sum_{j \geq 0} c_{ij} t^j e^{\lambda_i t}$

Distributed delay and Several rings

Examples: Elementary distributed delay

$$\theta_\lambda(t) = \begin{cases} \mathrm{e}^{\lambda t} & , \quad t \in [0, \vartheta] \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Its Laplace transform writes

$$\hat{\theta}_\lambda(s) = \frac{1 - \mathrm{e}^{-(s-\lambda)\vartheta}}{s - \lambda}$$

which is an entire function even in $s = \lambda$ where $\hat{\theta}_\lambda(\lambda) = \vartheta$, and is then BIBO–stable for any $\lambda \in \mathbb{C}$

We call $\theta_\lambda(\cdot)$ the elementary distributed delay

Distributed delay and Several rings

Lemma [Lu et al., 2010]

Any distributed delay can be decomposed into a finite sum of Laplace transforms of elementary distributed delays and its successive derivatives.

Distributed delay: Ring $\mathscr{G}$

The set of distributed delays is denoted by $\mathscr{G}$

The ring $\mathscr{P}_{\mathscr{E}}$

$$\mathscr{P}_{\mathscr{E}} = \mathscr{G} + \mathscr{D}$$

where $\mathscr{D}$ is the set of elements of the form $d(t) = \sum_{k=0}^{r} d_k \delta(t - k\vartheta)$ with $d_k \in \mathbb{R}$, $r \in \mathbb{N}$ being finite

Model matching problem

Part. 2 Model matching problem

Examples: $T(s)C(s) = T_m(s)$

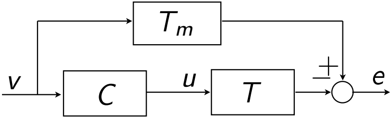
$T_m = \frac{1}{s+1},\ T(s) = \frac{s+2}{s+1},\ C(s) = ?$

$T_m = \frac{s+1}{s+2},\ T(s) = \frac{s-1}{s+1},\ C(s) = ?$

Model matching problem

Finite time model matching:

Exact model matching is obtained after a finite time interval



Model matching problem:

Let $T(s)$ be a given plant. Find a stable pre-compensator $C(s)$ such that the error on model matching

$$\hat{e}(s) = (T_m(s) - T(s)C(s))\hat{v}(s)$$

is the Laplace transform of a causal function with finite support

Model matching problem

error in the ring $\mathscr{P}_{\mathscr{E}} = \mathscr{G} + \mathscr{D}$

Let $e(t)$ be in the set $\mathscr{P}_{\mathscr{E}}$.Any element in $e \in \mathscr{P}_{\mathscr{E}}$ is of the form

$$e(t) = g(t) + \sum_{k=0}^{r} e_k \delta(t - k\vartheta)$$

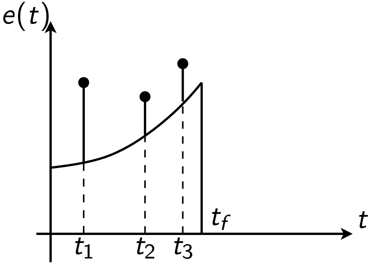
where $g \in \mathscr{G}$ is a distributed delay, and $e_k \in \mathbb{R}$, for $k = 0, \ldots, r$. In the Laplace domain, this is equivalent to say that any $\hat{e}(s) \in \hat{\mathscr{P}}_{\mathscr{E}}$ can be written like

$$\hat{e}(s) = \hat{g}(s) + \sum_{k=0}^{r} e_k \mathrm{e}^{-k\vartheta s}$$

Elements in $\hat{\mathscr{P}}_{\mathscr{E}}$ are proper and BIBO–stable, since they have a finite support

Model matching problem

error in the ring $\mathscr{P}_{\mathscr{E}}$



The time interval $[0, t_f]$ enclose the support of $e(t)$. Its length is related to the support of the distributed delay $g(t)$ and $r\vartheta$

Model matching for scalar linear systems

Model matching for scalar linear systems

Procedure

1. Take any coprime factorization (N, D) of $T(s)$ in $\mathbb{R}[s]$ and (N_m, D_m) of $T_m(s)$ in $\mathbb{R}[s]$.

$$T(s) = \frac{N(s)}{D(s)} \quad T_m(s) = \frac{N_m(s)}{D_m(s)}$$

2. Decompose the $N(s)$

$$N(s) = N_{\mathrm{s}}(s) N_{\mathrm{u}}(s)$$

3. Bezout equation in $\mathscr{G} + \mathbb{R}[s, \mathrm{e}^{-\vartheta s}]$

$$X(s) N_{\mathrm{u}}(s) + Y(s) D_m(s) = 1$$

Model matching for scalar linear systems

Procedure

4. Euclidean division

$$X(s)N_m(s) = Q(s)D_m(s) + P(s)$$

where $Q(s), P(s) \in \mathscr{G} + \mathbb{R}[s, \mathrm{e}^{-\vartheta s}]$, with $\deg_s P(s) < \deg_s D_m(s)$

5.

$$C(s) = D(s)N_s^{-1}P(s)D_m^{-1}(s)$$

and

$$\hat{e}(s) = Y(s)N_m(s) + N_{\mathrm{u}}(s)Q(s)$$

in $\hat{\mathscr{P}}_{\mathscr{E}}$

Model matching for scalar linear systems

Example 1: model matching example

Take

$$T(s) = \frac{s-1}{s(s+1)} \text{ and the model } T_{\mathrm{m}} = \frac{1}{s}$$

We have

$$N_{\mathrm{s}} = 1, N_m = 1, N_{\mathrm{u}} = s-1, D = s(s+1), D_m = s$$

Take

$$X(s) = \frac{1}{1-\mathrm{e}^{\vartheta}}\left[\frac{1-\mathrm{e}^{-(s-1)\vartheta}}{s-1}\right], Y(s) = \frac{-\mathrm{e}^{\vartheta}}{1-\mathrm{e}^{\vartheta}}\left[\frac{1-\mathrm{e}^{-s\vartheta}}{s}\right]$$

that satisfy $X(s)N_{\mathrm{u}}(s) + Y(s)D_m = 1$

Model matching for scalar linear systems

Example 1 (continued)

$$X(s)N_m(s) = Q(s)D_m(s) + P(s)$$

$Q = 0$ and $P(s) = \frac{1}{1-e^{\vartheta}}\left[\frac{1-e^{-(s-1)\vartheta}}{s-1}\right]$. A precompensator takes the form

$$C(s) = \frac{1}{1-e^{\vartheta}}(s+1)\left(\frac{1-e^{-(s-1)\vartheta}}{s-1}\right)$$

while the error is

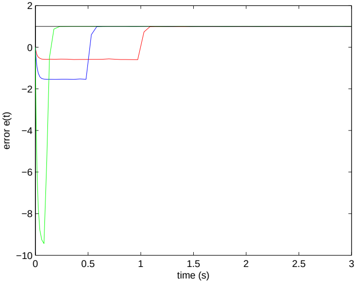
$$\hat{e}(s) = \frac{-e^{\vartheta}}{1-e^{\vartheta}}\left[\frac{1-e^{-s\vartheta}}{s}\right]$$

Model matching for scalar linear systems

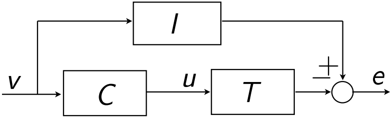
Example 1 (continued)

In the time domain, this error is

$$e(t) = \begin{cases} \frac{-e^{\vartheta}}{1-e^{\vartheta}} & , \quad t \in [0, \vartheta] \\ 0 & , \quad \text{elsewhere} \end{cases}$$



Model matching for scalar linear systems



Stable inversion problem:

Let $T(s)$ be a given plant. Find a stable pre-compensator $C(s)$ such that the error on model matching

$$\hat{e}(s) = (1 - T(s)C(s))\hat{v}(s)$$

is the Laplace transform of a causal function with finite support

Model matching for scalar linear systems

Example 2: stable inversion

Consider the plant

$$T(s) = \frac{s-1}{s+1}$$

where $N_{\mathrm{u}}(s) = s - 1$, $N_{\mathrm{s}}(s) = 1$ and $D(s) = s + 1$. Take for instance

$$X(s) = \frac{1 - \mathrm{e}^{-(s-1)\vartheta}}{s-1} \in \hat{\mathscr{G}} \qquad Y(s) = \mathrm{e}^{-(s-1)\vartheta}$$

Model matching for scalar linear systems

Example 2 (continued)

The construction of the pre-compensator

$$C(s) = D(s)N_s^{-1}X(s)$$

A precompensator that solves the finite time stable inversion is

$$C(s) = (s+1)\frac{1 - \mathrm{e}^{-(s-1)\vartheta}}{s-1}$$

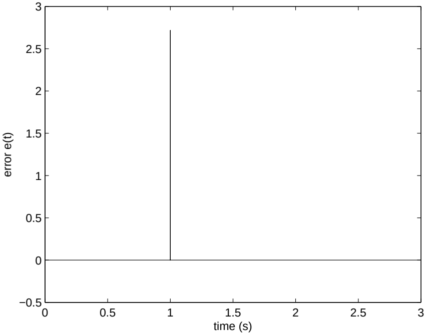
The error is $\hat{e}(s) = \mathrm{e}^{-(s-1)\vartheta}$.

Model matching for scalar linear systems

Example 2 (continued)

In the time domain, this inverse error is

$$e(t) = \mathrm{e}^{\vartheta}\,\delta(t-\vartheta)$$



Model matching for multivariable linear systems

Part. 3 Model matching for multivariable linear systems

1. Procedure
2. example for model matching
3. exmple for an application: stable inversion

Model matching for multivariable linear systems

Procedure

1. Take any coprime factorization (N, d) of $T(s) \in \mathbb{R}^{m\times n}(s)$ and (N_m, d_m) of $T_m(s) \in \mathbb{R}^{m\times q}(s)$, then

$$T(s) = \frac{N(s)}{d(s)}, \qquad T_m(s) = \frac{N_m(s)}{d_m(s)}$$

where $d(s), d_m(s) \in \mathbb{R}[s]$, $N(s) \in \mathbb{R}^{m\times n}[s]$, $N_m \in \mathbb{R}^{m\times q}[s]$

Model matching for multivariable linear systems

Procedure

2. Take the Smith form of the matrix $N(s)$. We have $N(s) = U_1 \Lambda U_2$ where $U_1 \in \mathbb{R}^{m\times m}[s]$, $U_2 \in \mathbb{R}^{n\times n}$ are unimodular matrices, Λ is the unique diagonal matrix in $\mathbb{R}^{m\times n}[s]$ of the form

$$\Lambda = \left[\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & \mathbb{O} \\ & & \lambda_r & \\ \hline & \mathbb{O} & & \mathbb{O} \end{array}\right]_{m\times n} .$$

Model matching for multivariable linear systems

Procedure

3. We decompose Λ by $\Lambda = \Lambda_u \Lambda_s$, where

$$\Lambda_u = \left[\begin{array}{ccc|c} \lambda_{1u} & & & \\ & \ddots & & \mathbb{O} \\ & & \lambda_{ru} & \\ \hline & \mathbb{O} & & \mathbb{O} \end{array}\right]_{m \times n},$$

$$\Lambda_s = \left[\begin{array}{ccc|c} \lambda_{1s} & & & \\ & \ddots & & \mathbb{O} \\ & & \lambda_{rs} & \\ \hline & \mathbb{O} & & R(s) \end{array}\right]_{n \times n}.$$

Then $N(s) = U_1 \Lambda_u \Lambda_s U_2$

Model matching for multivariable linear systems

Procedure

4. Let $V(s) = U_1\Lambda_u$, $D_m = d_m \cdot I_m$. If $V(s)$ and D_m are not left coprime, there exist $\tilde{V}(s) \in \mathbb{R}^{m\times n}[s]$ and $F(s) \in \mathbb{R}^{n\times n}[s]$ such that $V(s) = \tilde{V}(s)F(s)$, and $\tilde{V}$ and D_m are left coprime. Then we have

$$\tilde{V}(s)X(s) + D_m Y(s) = I_m$$

5. We divide $X(s)N_m(s)$ by $d_m(s)\cdot \mathrm{I}_q$. We see that there exist $Q(s)$ and $P(s)$ in $\mathscr{E}^{n\times q}$ such that

$$X(s)N_m(s) = Q(s)d_m(s)\cdot \mathrm{I}_q + P(s),$$

with $P \cdot d_m^{-1}$ a strictly proper matrix.

Model matching for multivariable linear systems

6. Define the pre-compensator

$$C(s) = d(s)\left(\Lambda_s(s)U_2(s)\right)^{-1} F^{-1}(s)P(s)d_m^{-1}(s)$$

is stable and proper (or strictly proper), and the error is

$$E(s) = T_m(s) - T(s)C(s) = Y(s)N_m(s) + \tilde{V}(s)Q(s) \in \hat{\mathscr{P}}_{\mathscr{E}}$$

Model matching for multivariable linear systems

Example 3: model matching example

Consider the plant

$$T(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} \end{bmatrix},$$

and the model $T_m(s) = \frac{s+2}{s+1}$. We have

$$N(s) = [s-1 \;\; 1], \;\; d = s+1,$$

$$N_m(s) = s+2, \;\; d_m = s+1.$$

We decompose

$$N(s) = U_1(s)\Lambda_u(s)\Lambda_s(s)U_2(s),$$

where $U_1(s) = \mathrm{I}$, $U_2(s) = \begin{bmatrix} s-1 & 1 \\ 1 & 0 \end{bmatrix}$, $\Lambda_u(s) = [1 \;\; 0]$ $\Lambda_s(s) = \mathrm{I}_2$.

Model matching for multivariable linear systems

Example 3 (continued)

Take

$$X(s) = \begin{bmatrix} \mathrm{e}^{-(s+1)\vartheta} \\ 0 \end{bmatrix} \qquad Y(s) = \frac{1 - \mathrm{e}^{-(s+1)\vartheta}}{s+1},$$

such that $V(s)X(s) + D_m(s)Y(s) = \mathrm{I}$, where $V(s) = U_1(s)\Lambda_u(s)$. The Division yields

$$Q(s) = \begin{bmatrix} \mathrm{e}^{-(s+1)\vartheta} \\ 0 \end{bmatrix}, \quad P(s) = \begin{bmatrix} \mathrm{e}^{-(s+1)\vartheta} \\ 0 \end{bmatrix}.$$

The precompensator is the form

$$C(s) = \begin{bmatrix} 0 \\ \mathrm{e}^{-(s+1)\vartheta} \end{bmatrix},$$

Model matching for multivariable linear systems

Example 3 (continued)

and the model matching error is

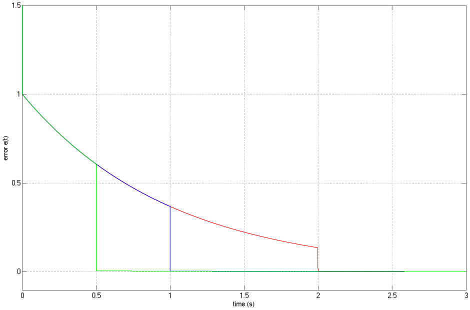
$$E(s) = 1 + \frac{1 - \mathrm{e}^{-(s+1)\vartheta}}{s+1}.$$

In the time domain the error is

$$e(t) = \begin{cases} \delta(t) + \mathrm{e}^{-t}, & t \in [0, \vartheta] \\ 0, & \text{elsewhere} \end{cases} .$$

Model matching for multivariable linear systems

Example 3 (continued)



Model matching for multivariable linear systems

Example 4: stable inversion example

We consider

$$T = \begin{bmatrix} \frac{s-1}{s+1} \\ \frac{1}{s+1} \end{bmatrix}$$

with $N(s) = \begin{bmatrix} s-1 \\ 1 \end{bmatrix}$, $d(s) = s+1$. Decomposed $N(s) = N = U_1 \Lambda_u \Lambda_s U_2$, we have $U_1 = \begin{bmatrix} s-1 & 1 \\ 1 & 0 \end{bmatrix}$, $U_2 = \mathrm{I}$, $\Lambda_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\Lambda_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $V(s) = \begin{bmatrix} s-1 \\ 1 \end{bmatrix}$.

Model matching for multivariable linear systems

Example 4 (continued)

Take

$$X(s) = \begin{bmatrix} \frac{1-\mathrm{e}^{-(s-1)\vartheta}}{s-1} & 0 \end{bmatrix} \in \mathscr{P}_{\mathscr{E}}^{1\times 2}.$$

$$Y(s) = \begin{bmatrix} \mathrm{e}^{-(s-1)\vartheta} & 1 \\ 1 - \frac{1-\mathrm{e}^{-(s-1)\vartheta}}{s-1} & 1 \end{bmatrix} \in \mathscr{P}_{\mathscr{E}}^{2\times 2}$$

such that

$$V(s)X(s) + Y(s) = I$$

Model matching for multivariable linear systems

Example 4 (continued)

The construction of the pre-compensator

$$C(s) = d(s)(\Lambda_s U_2(s))^{-1} X(s)$$

The proper and stable precompensator

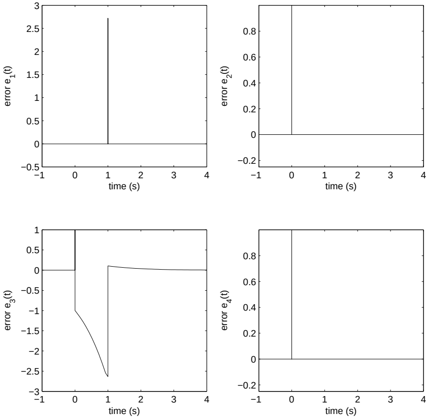
$$C(s) = \begin{bmatrix} (s+1)\dfrac{1-\mathrm{e}^{-(s-1)\vartheta}}{s-1} & 0 \end{bmatrix} \in \mathscr{E}^{1\times 2},$$

and the inversion error is

$$E(s) = \begin{bmatrix} \mathrm{e}^{-(s-1)\vartheta} & 1 \\ 1 - \frac{1-\mathrm{e}^{-(s-1)\vartheta}}{s-1} & 1 \end{bmatrix} \in \mathscr{P}_{\mathscr{E}}^{2\times 2}$$

Model matching for multivariable linear systems

Example 4 (continued)



Conclusions

- Finite time model matching
- Algorithm of the pre-compensator
- An application of finite time model matching: finite time stable inversion

Thank you for your attention !