About the stability of sampled-data systems with non-uniform sampling

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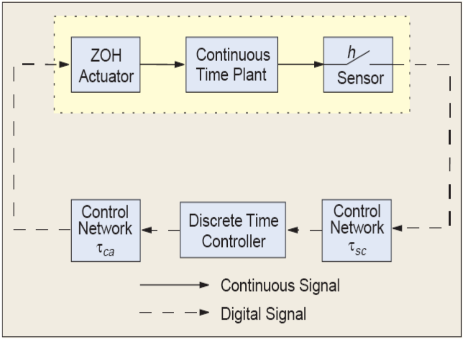
Outline

Motivating problem : Digital control

Classical control loop

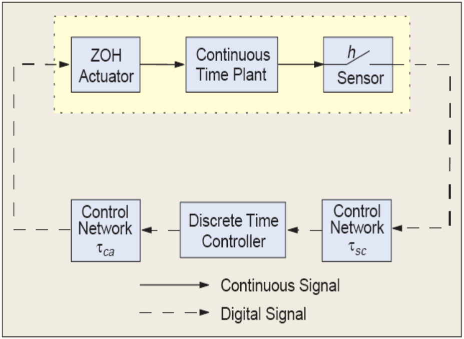


Ideal Hypothesis :

- Sampling and actuation are periodic and synchronous

Motivating problem : Digital control

Classical control loop

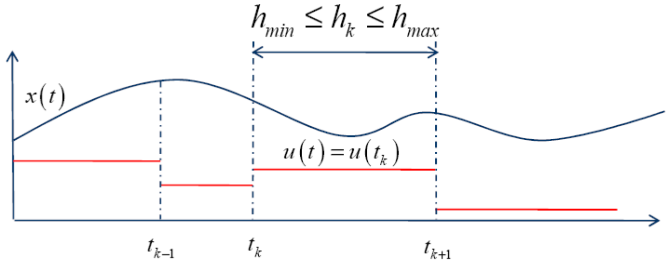


Real-time problem : the system is affected by timing problems

- sampling jitter (sensor, multitasking processors, packet dropouts in communication channnels)
- unknown time varying delays (not adressed here)

(Wittenmark, Nilsson, Torngren, 1995)

Problem Formulation



Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \ \forall t \in \mathbb{R}^+$$

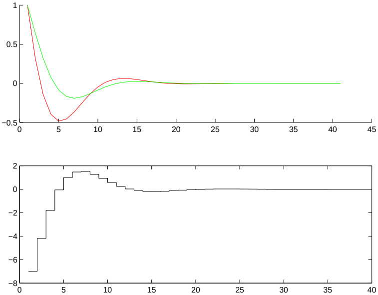
with a sampled-data control :

$$u(t) = Kx(t_k), \ \forall t \in [t_k, t_{k+1})$$

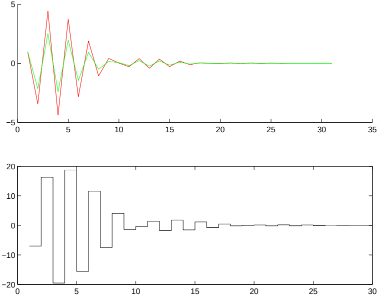
Problem : is the system robust to jitter ?

Sampling jitter example (Zhang,2001)

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \;\; h_k \in \{T_1, T_2\}$$



$h = 0.18s$

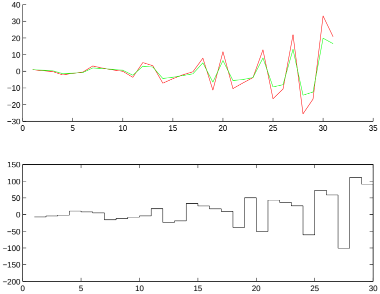


$h = 0.58s$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}; \quad K = -[1 \;\; 6]$$

Sampling jitter example (Zhang,2001) $\Rightarrow$ instability

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \; h_k \in \{T_1, T_2\}$$



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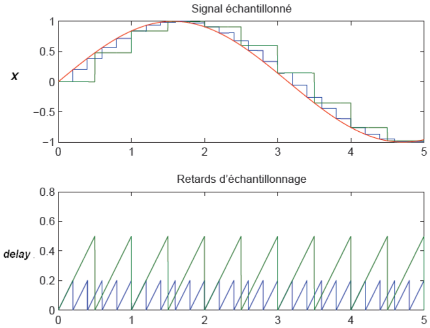
Open problem : provide tools for robust stability and performance analyzis !

Existing work - Continuous-time : time delay approach



$$u(t) = Kx(t_k) = Kx(t-\tau) \text{ with } \tau = t - t_k,\ 0 < \tau_k < h_{max}$$

Existing work - Continuous-time

- Fridman et al, 2004 (input delay appraoch)
- Mirkin, 2007 (robust control equivalent)
- Hespanha, 2008 (impulsive delay diff. eq.)

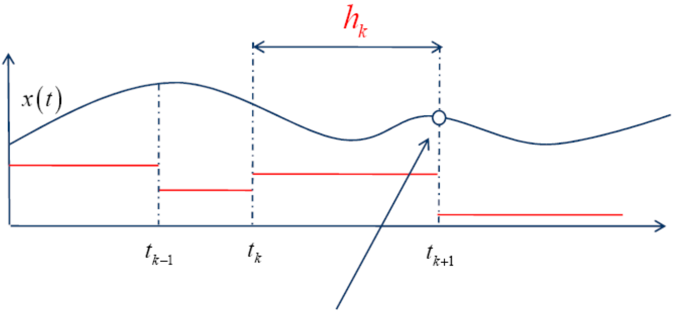
Advantage :

- Directly extend to performance study (decay rate)
- Take into account the inter-sampling behaviour

Inconvenient :

- Do not take into account the sawtooth form of the delay (conservatism)

Existing work - Discrete-time (LPV model)



$$x(t_{k+1}) = e^{(t_{k+1}-t_k)A} x(t_k) + \int_0^{(t_{k+1}-t_k)} e^{sA} ds B u(t_{k+1})$$

$$\Longrightarrow \quad x_{k+1} = \Lambda(h_k) x(t_k)$$

Existing work - Discrete-time (LPV model)

- Sala, 2004 ; Boyd,2008 (gridding approach)
- Hetel, 2006, 2009 (convex embedding)
- Fujioka, 2007 (gridding + robust control)

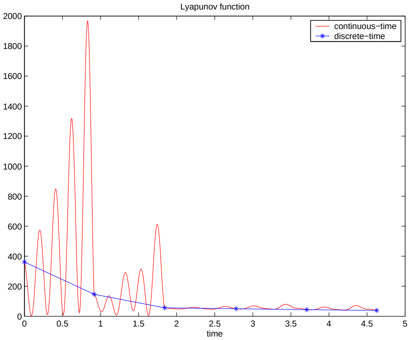
Advantage :

- Implicitly take into account the sawtooth form of the delay (less conservative)

Inconvenient :

- Not numerically efficient when the minimum sampling interval goes to zero
- Ignore the inter-sampling behaviour

Discrete-time problem (inter-sampling behaviour)



Evolution of a Lyapunov function $V(x) = x^T P x$

- Striclty decreasing at $t = t_k$
  (sufficient condition for stability analyzis)
- Increasing in between the sampling times
  (false evaluation of control performance)

Goal

Provide a continuous-time method that takes into account the sawtooth form of the delay
(advantage of discrete-time methods for conservatism reduction)

Exponential Stability

- how fast the norm of the state vector converges to zero. $\alpha > 0$

$$\|x(t)\| \leq e^{-\alpha t} c \|x(0)\|, \forall t > 0$$

- for a given candidate Lyapunov function $V(x)$, if

$$\frac{dV(x)}{dt} < -2\alpha V(x), \; \forall x \neq 0$$

Case of quadratic Lyapunov functions

For $\frac{dx(t)}{dt} = Ax(t) + BKx(t - \tau(t)),\ \tau(t) := t - t_k,\ \forall t \in [t_k, t_{k+1})$

and $V(x) = x^T P x$

- Derivative of Lyapunov function

$$\frac{dV(x)}{dt} = 2x^T(t)P\left(Ax(t) + BKx(t - \tau)\right).$$

- Sawtooth evolution of delay can be introduced by using the integration operator $\Lambda(\cdot)$ used for the discrete-time model :

$$x(t) = \Lambda(t - t_k)\, x(t_k),\ \forall t \in [t_k, t_{k+1})$$

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$$x(t) = \Lambda(\tau)\, x(t - \tau),\ \forall t \in [t_k, t_{k+1})$$

Estimation of the decay rate

$$\alpha > 0, \;\; P \succ 0$$

$$\left(K^T B^T + \Lambda^T(\tau) A^T\right) P \Lambda(\tau) + \Lambda^T(\tau) P \left(A\Lambda(\tau) + BK\right) \\ \prec -2\alpha \Lambda^T(\tau) P \Lambda(\tau),$$

$$\forall \tau \in [0, h_{max}).$$

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- Problem : infinite number of LMI conditions !

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$$\forall \tau \in [0, h_{max}).$$

- Problem : infinite number of LMI conditions !
- Solution : convex embedding with finite number of generators

Exponential uncertainty

- Integration operator

$$\begin{aligned}\Lambda(\rho) &= e^{\rho A} + \int_0^{\rho} e^{sA} ds BK \\ &= I + \int_0^{\rho} e^{sA} ds \, (A + BK)\end{aligned}$$

- Exponential uncertainty :

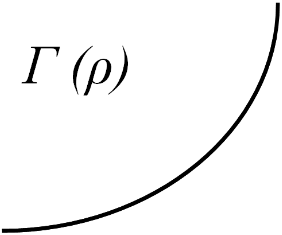
$$\Gamma(\rho) = \int_0^{\rho} e^{sA} ds, \ \ \rho \in [0, h_{max}).$$

- Use a classical representation from the robust control point of view !

Exponential uncertainty - Parametric uncertainty representation

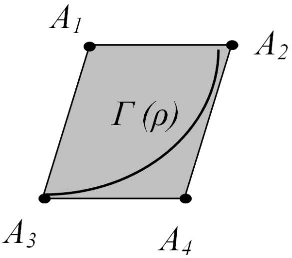
$$\Gamma(\rho) = \int_0^{\rho(k)} e^{As} ds$$

$$0 < \rho < h_{max}$$



Treat the term like a parametric uncertainty

Exponential uncertainty - Convex Polytope



$$\Gamma(\rho) = \sum_{i=1}^{N} \mu_i A_i$$

$$\sum_{i=1}^{N} \mu_i = 1, \ \mu_i > 0, \ \forall i = 1, \ldots, N$$

(Hetel,Trans. Autom. Contr. 2006)
(Cloosterman, Trans. Autom. Contr. 2009),
(Olaru, IFAC World Congress 2007),

Jordan normal form

For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

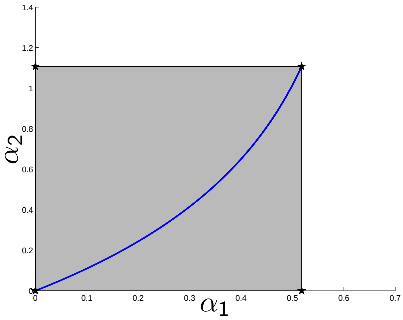
$$\Gamma(\rho) = \begin{pmatrix} \alpha_1(\rho) & 0 \\ 0 & \alpha_2(\rho) \end{pmatrix}$$

with

$$\alpha_i(\rho) = \int_0^\rho e^{\lambda_i s} ds$$

vertex = max or min $\alpha_i(\rho)$

$\lambda_1 = -1.5, \lambda_2 = 0.2$



$$\Gamma(\rho) = \sum_{j=0}^{2^n} \mu_j A_j$$

Jordan normal form + gridding

For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\Gamma(\rho) = \begin{pmatrix} \alpha_1(\rho) & 0 \\ 0 & \alpha_2(\rho) \end{pmatrix}$$

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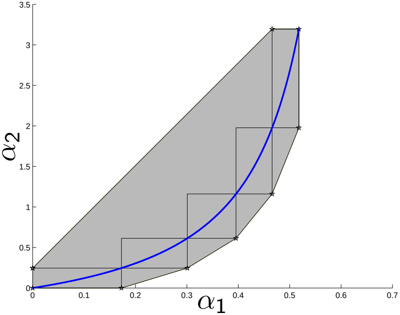
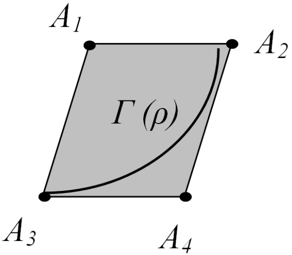


$$\Gamma(\rho) = \sum_{j=0}^{5\times 2^n} \mu_j A_j$$

Tractable LMI conditions



$$\Gamma(\rho) = \sum_{i=1}^{N} \mu_j A_j$$

$$\sum_{j=1}^{N} \mu_j = 1, \ \mu_j > 0, \ \forall j = 1, \ldots, N$$

Stability conditions : $P = P^T \succ 0$

$$\begin{pmatrix} A^T P + PA + G_1 + G_1^T + \alpha P & PBK - G_1 A_j + G_2^T \\ K^T B^T P - A_j^T G_1^T + G_2 & -G_2 A_j - A_j^T G_2^T \end{pmatrix} \prec \mathbf{0},$$

$$\forall l = 1, \ldots, N.$$

Numerical example

Consider a continuous-time system described by :

$$A = \begin{pmatrix} 1 & 15 \\ -15 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

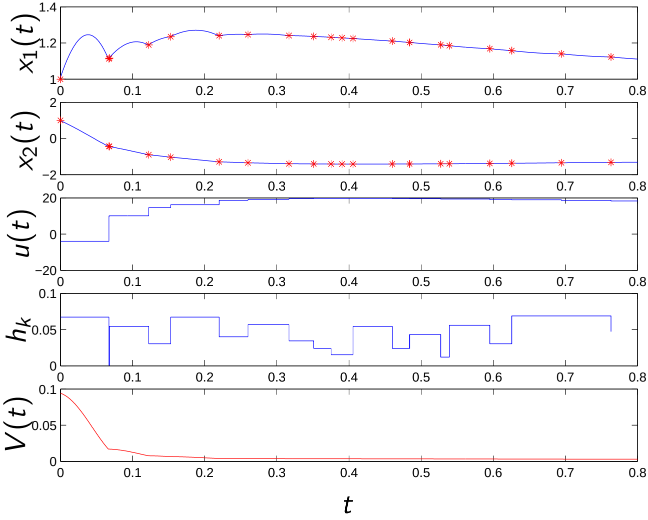
- $\lambda(A) = 1 \pm 15i$
- K - obtained by pole placement :

$$\lambda(A + BK) = -1 \pm i$$

Stability analyzis comparison :

- (Mirkin, TAC 2007) : $h \in [0, 0.014]$
- (Naghshtabrizi, Hespanha, Teel, SCL 2008) :$h \in [0, 0.033]$
- (Fujioka, Automatica 2009) : $h \in [0, 0.07]$
- continuous-time convex embedding : $h \in [0, 0.09]$

Numerical example



Conclusion and Perspective

- Robustess to sampling jitter
- Provide robust methods for stability
- Show how to reduce the conservatism of stability analyzis by taking into account the sawtooth form of the delay
- Perspective : control design